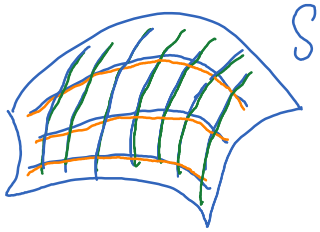
Lecture 37

Wednesday, April 6, 2022 9:05 AM

* Prayer
* Spiritual thought

Tangent plane



$r(u,v) = (x(u,v), y(u,v), z(u,v))$

Green curve: fix u, move v.

Orange curve: fix v, move u.

r_u and r_v are two tangent vectors

Tangent vector along the green curve is r_v.

" " orange " " r_u.

Normal vector to the tangent plane is $r_u \times r_v$.

From here we can find the equation of the tangent plane.

Ex S: $r(u,v) = (u+2v, u^2 - uv, uv+v)$

Find the tangent plane at point $(4,2,3)$.

$$\begin{cases} u+2v=4 \\ u^2-uv=2 \\ uv+v=3 \end{cases}$$

$u+2v=4 \longrightarrow u+2(5-u^2)=4 \Rightarrow u=2$ or $u=-\frac{3}{2}$

$u^2-uv=2,\ uv+v=3 \longrightarrow u^2+v=5 \longrightarrow v=5-u^2$

Take $u=2 \longrightarrow v=1$

$$r_u = (1,\ 2u-v,\ v) \longrightarrow r_u(2,1) = (1,3,1)$$

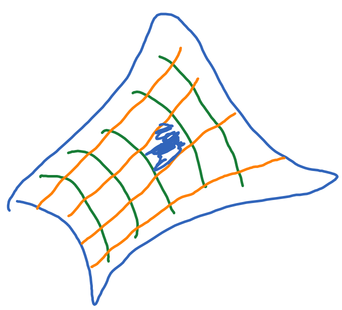
$$r_v = (2,\ -u,\ u+1) \longrightarrow r_v(2,1) = (2,-2,3)$$

$$r_u \times r_v = (11,-1,-8).$$

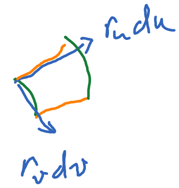
eq of tangent plane : $11(x-4) + (-1)(y-2) + (-8)(z-3) = 0.$

<u>Surface integral</u>

$$\underbrace{\iint_S f\,dS}_{\text{used to find mass/area}} \quad \text{and} \quad \underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{used to find flux.}}$$



dS = area of a cell $\approx |r_u du \times r_v dv|$



$= |r_u \times r_v|\,du\,dv$

Thus, $$\iint_S f\,dS = \iint_R f(r(u,v))\,|r_u \times r_v|\,dA$$

<u>Ex.</u> $\iint_S z\,dS$ where S is the upper half of the unit sphere.

$$S: \begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad \begin{matrix} 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$r_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

$$r_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

$$r_\phi \times r_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta)$$

$$= (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi)$$

$$|r_\phi \times r_\theta|^2 = \sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi$$

$$= \sin^4\phi + \cos^2\phi\sin^2\phi = \sin^2\phi$$

$$\longrightarrow |r_\phi \times r_\theta| = \sin\phi$$

$$\iint_S z\,dS = \iint_S \cos\phi\,|r_\phi \times r_\theta|\,dS = \int_0^{2\pi}\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\,d\theta = \cdots$$

Note: for sphere of radius ρ, $|r_\phi \times r_\theta| = \rho^2\sin\phi$, which is the same as the Jacobian.

Ex Find surface area of the cone $z^2 = x^2 + y^2$ between $z = 0$ and $z = 1$.

Two ways to parametrize this cone:

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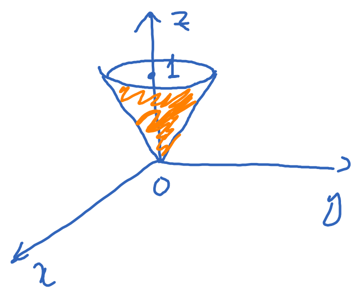

Two ways to parametrize this cone:

$$(\mathrm{I}): \begin{cases} x = u \\ y = v \\ z = \sqrt{u^2+v^2} \end{cases}, (u,v) \in \text{unit disk}$$

$$(\mathrm{II}): \begin{cases} x = v\cos u \\ y = v\sin u \\ z = v \end{cases} \quad \begin{matrix} 0 \le v \le 1 \\ 0 \le u \le 2\pi \end{matrix}$$

Using parametrization (I)

$$r_u = (x_u, y_u, z_u) = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right)$$

$$r_v = (x_v, y_v, z_v) = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right)$$

$$r_u \times r_v = \left(\frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1\right)$$

$$|r_u \times r_v| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{2}$$

$$\text{Area of surface} = \iint_{\text{disk}} |r_u \times r_v|\, dA = \iint_{\text{disk}} \sqrt{2}\, dA = \sqrt{2}\ \text{area(disk)} = \sqrt{2}\pi$$

Using parametrization (II)

$$r = (v\cos u, v\sin u, v)$$

$$\left.\begin{aligned} r_u &= (-v\sin u, v\cos u, 0) \\ r_v &= (\cos u, \sin u, 1) \end{aligned}\right\} \quad r_u \times r_v = (v\cos u, v\sin u, -v)$$

$$|r_u \times r_v| = \sqrt{v^2\cos^2 u + v^2 \sin^2 u + v^2} = \sqrt{2v^2} = v\sqrt{2}.$$

area of S is

$$\int_0^1 \int_0^{2\pi} |r_u \times r_v| \, du\, dv = \int_0^1 \int_0^{2\pi} v\sqrt{2}\, du\, dv = \pi\sqrt{2}$$